ARTICLE NO. 78-1334R

Angular Motion of Spinning Almost-Symmetric Missiles

Charles H. Murphy*

Ballistic Research Laboratory, Armament Research and Development Command, Aberdeen Proving Ground, Md.

An almost-symmetric missile is a missile whose zero-spin pitch and yaw frequencies are "nearly" equal. The angular motion of a spinning almost-symmetric missile can be described by five rotating modal vectors. This pentacyclic motion reduces to the usual tricyclic motion when the frequencies are equal. For spins that are not near zero or near the region between pitch and yaw resonance, the motion is well approximated by the tricyclic motion of a symmetric missile with average moment and force coefficients. For spins in the resonance region, the motion is usually exponentially undamped and four of the modal amplitudes may be comparable in size. For spin near zero, all five modes may be comparable in size. Good approximate formulas for the size of the additional two modal vectors are given in terms of the amount of asymmetry.

Nomenclature

A $= k_t^{-2}(C^*_{m_0} + iC^*_{n_0})$
a $= (f\phi')^2/2 - M_r$
b $= M_r^2 - \hat{M}^2$
C_D $= \dfrac{\text{drag force}}{\rho S V^2/2}$
C_l, C_m, C_n = missile fixed components of $\left(\dfrac{\text{aerodynamic moment}}{\rho S l V^2/2}\right)$
C_X, C_Y, C_Z = missile-fixed components of $\left(\dfrac{\text{aerodynamic force}}{\rho S V^2/2}\right)$
C_1, C_2 $= \omega_j^2 - f\phi'\omega_j + M_r,\ j=1,2$
$C_{(\ )}, \hat{C}_{(\ )}$ see Table 4 for all C definitions not given above
f $= 2 - (I_x/I_t)$
g_1 $= -\omega_{2S}[2f\phi' M_r(\omega_{1S} - \omega_{2S})]^{-1}$
g_2 $= \omega_{1S}[2f\phi' M_r(\omega_{1S} - \omega_{2S})]^{-1}$
H $= C^*_{N_\alpha} - 2C^*_D - k_t^{-2}(C^*_{M_q} + C^*_{M_{\dot\alpha}})$
$\hat{H}$ $= \hat{C}^*_{N_\alpha} + k_t^{-2}(\hat{C}^*_{M_q} + \hat{C}^*_{M_{\dot\alpha}})$
I_t = transverse moment of inertia $= I_y = I_z$
I_x = axial moment of inertia
K_j = absolute value of k_j
K_{jR} = absolute value of k_{jR}
k_j = complex yaw modes, $j = 1$-5
k_{jR} = complex yaw modes in the resonance region, $j=1,2$
k_t^2 $= I_t/ml^2$
l = reference length
M $= k_t^{-2}C^*_{M_\alpha}$
$\hat{M}$ $= k_t^{-2}\hat{C}^*_{M_\alpha}$
M_r $= M + (f-1)(\phi')^2$
M_S $= P(C^*_{N_\alpha} - C^*_D) + k_t^{-2}C^*_{MS_\alpha}$
$\hat{M}_S$ $= f\phi'\hat{C}^*_{N_\alpha}$
m = mass
P $= (I_x/I_t)\phi'$
p = spin rate, rad/s
q, r = transverse angular velocity components in the missile-fixed system
S = reference area
s = dimensionless arclength along the trajectory
t = time
$u, v, w,$ = missile-fixed components of the velocity
V = magnitude of the velocity
X, Y, Z = missile-fixed axes in which the X axis is along the principal axis of inertia nearest to the flight direction
α = angle of attack $= \tan^{-1}(w/u)$
β = angle of sideslip $= \sin^{-1}(v/V)$
δ $= \begin{cases} 1 \text{ if } \phi' > \phi'_\alpha \\ -1 \text{ if } 0 \le \phi' < \phi'_\beta \end{cases}$
ϵ = any positive number small in comparison with $\phi'_\alpha - \phi'_\beta$
λ = exponential damping coefficient in the resonance-region solution of the yaw equation
λ_1, λ_2 = exponential damping coefficients in the general solution of the yaw equation
λ_{jS} $= -(H\phi'_j - M_S)/(2\phi'_j - P),\ \ j=1,2$ (the symmetric-missile value of λ_j)
μ $= (q + ir)l/V$
ζ = missile-fixed complex angle of attack $= (v + iw)/V$
$\tilde{\xi}$ = aeroballistic complex angle of attack $= \xi e^{i\phi}$
ρ = air density
ϕ = roll angle $= \phi' s$
ϕ_{jR} = argument of k_{jR}
ϕ_{j0} = argument of k_j
ϕ' = spin rate, rad/cal $= pl/v$
ϕ'_j = jth yaw arm frequency in the aeroballistic system, $j=1,2$
ϕ'_{jS} $= [P \pm \sqrt{P^2 - 4M}]/2, j=1,2$ (the symmetric-missile value of ϕ'_j)
ϕ'_α = upper boundary of the resonance spin region
ϕ'_β = lower boundary of the resonance spin region
ω_j = jth yaw arm turning rate in the missile-fixed system, $j=1,2$
ω_α = pitch frequency for zero spin $= (-k_t^{-2}C^*_{m_\alpha})^{1/2}$
ω_β = yaw frequency for zero spin $= (k_t^{-2}C^*_{n_\beta})^{1/2}$

Presented as Paper 78-1334 at the AIAA Atmospheric Flight Mechanics Conference, Palo Alto, Calif., Aug. 7-9, 1978; submitted Sept. 25, 1978; revision received March 12, 1979.

Index categories: LV/M Dynamics and Control; Entry Vehicle Dynamics and Control.

*Chief, Launch and Flight Division. Associate Fellow AIAA.

Superscripts

$(\dot{\ })$ $= d(\ )/dt$
$(\ )'$ $= d(\ )/ds = (\dot{\ })l/V$
$(\ )^*$ $= (\rho Sl/2m)(\ )$
$(\bar{\ })$ = complex conjugate of ( )

I. Introduction

IN 1970, Ward and Mansfield[1] made some hypersonic damping-in-pitch measurements of 9-deg cones by the free oscillation technique. Since a spherical air-bearing support was used, the models were free to spin as well as pitch and yaw. An apparently minor tip asymmetry was induced by a planar cut 40 deg off the X-Z plane. The projected frontal area of the cut was approximately 0.5% of the base area.

The free oscillations were then fitted by the Nicolaides tricyclic theory.[2] This theory assumes that the sole effect of small asymmetry is to add a missile-fixed constant-amplitude trim moment term to the moment expression for the unmodified symmetric cone. A missile that satisfies this assumption is said to have a slight configurational asymmetry. It was found, however, that the measured motion cannot be reasonably represented by the tricyclic theory. In 1972, Walchner, Sawyer, and Yelmgren[3] showed that the apparently very small asymmetry used in the damping tests was sufficient to make the pitch and yaw frequencies unequal and thus a more complex mathematical moment assumption is required to analyze the free oscillation data. References 4 and 5 discuss two other cases of apparently symmetric missiles whose motions seem to be poorly described by either the epicyclic motion performed by symmetric missiles or the tricyclic motion performed by slightly asymmetric missiles.

The effect of constant axial spin on the stability of aircraft was first considered by Phillips.[6] Recently, Hodapp[7,8] has extended this work to unsymmetric re-entry vehicles in a most elegant fashion. Both Phillips and Hodapp make use of missile-fixed coordinates. These coordinates are convenient if the missile is not spinning or if on-board instrumentation is used. For the three experiments cited earlier, external photographic instrumentation was used and nonrolling aeroballistic axes are much more convenient. A second difficulty of the Phillips and Hodapp papers is that the general pitching and yawing motion of unsymmetric missiles is described by five modal vectors and the results are given in a form in which it is difficult to see how the five vectors reduce to the three modal vectors appropriate to symmetric motion with trim.

In this paper, we make use of the complex variables that considerably simplify the analysis of symmetric-missile motion.[9] These variables allow us to move easily from missile-fixed axes to nonspinning axes. They also allow us to obtain very simple expressions for an "almost" symmetric missile. All results are obtained for constant spin.

In the next four sections, the analysis is limited to a linear yaw moment and a linear pitch moment with C_{m_α} not equal to $-C_{n_\beta}$. Since the pitch and the yaw frequencies are different for zero spin, there is a *resonance region* for spin where the spin is betweeen these frequencies. After deriving exact relations for the two frequencies of the combined pitching and yawing motion for a spinning missile, we obtain approximations for an "almost-symmetric" missile. An almost symmetric missile is one whose zero-spin pitch frequency is very near its zero-spin yaw frequency. For these approximations, three regions of spin are considered separately: spin near zero, spin near resonance and spin far from zero spin and resonance spin. For the third region, it is shown that in the aeroballistic system two of the frequencies are approximated by the fast and slow rates $\dot{\phi}_{1S}$, $\dot{\phi}_{2S}$ appropriate to a symmetric missile with a static moment derivative

$$C_{M_\alpha} = (C_{m_\alpha} - C_{n_\beta})/2 \tag{1}$$

Two other modes exist with frequencies $2p - \dot{\phi}_{1S}$, $2p - \dot{\phi}_{2S}$. The amplitude of these modes is shown to approach zero as $C_{m_\alpha} \to -C_{n_\beta}$.

Near resonance, the fast mode and its associated frequency $2p - \dot{\phi}_1$ both approach the spin rate, p. For spin rates in the resonance zone between the pitch and yaw frequencies, both $\dot{\phi}_1$ and $2p - \dot{\phi}_1$ become equal to the spin rate and one mode is exponentially damped while the other is exponentially undamped.

Finally, the effects of linear aerodynamic damping and aerodynamic forces are studied. Their major effect is to reduce the region of spin rates for which exponential undamping exists. Outside the resonance region, the two damping rates are very similar to those for a completely symmetric missile.

II. Equations of Motion

The usual coordinates to describe the angular motion of a rigid body are body-fixed coordinates. For the case of a symmetric missile, the aeroballistic axes, which pitch and yaw with the missile but have a zero spin rate, are more convenient. In this study of an almost-symmetric missile, we will use missile fixed axes to obtain our results and then transform the results to the aeroballistic axes.

The X axis is fixed in the body along the principal axis of inertia nearest to the desired flight direction. Since we retain the assumption of mass rotational symmetry, the transverse moments of inertia are equal and for any roll orientation of the Y and Z axes, these axes will also be along principal axes of inertia. The actual orientation of these axes will be selected to simplify the aerodynamic moment. (The effect of mass asymmetry is considered in Ref. 10.)

The complex notation that is so helpful for a symmetric missile can now be introduced. The complex angle-of-attack and transverse angular velocity have the usual definitions:

$$\xi = (v + iw)/V \doteq \beta + i\alpha \tag{2}$$

$$\mu = (q + ir)l/V \tag{3}$$

The equations of motion can be written in terms of the aerodynamic force and moment and the derivatives of the linear and angular momentum. The small effect of gravity on the angular motion will be neglected. If the Z-components of the linear and angular momentum equations are multiplied by i and added to the corresponding Y-components and the independent variable changed from time t to dimensionless arclength s, the following two equations can be obtained:

$$\xi' + i\phi'\xi - i\mu = (C_Y^* + iC_Z^*) + C_D^*\xi \tag{4}$$

$$\mu' + i(\phi' - P)\mu = k_t^{-2}(C_m^* + iC_n^*) + C_D^*\mu \tag{5}$$

The most general moment components that are linear in α and β are

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_\beta}\beta \tag{6}$$

$$C_n = C_{n_0} + C_{n_\alpha}\alpha + C_{n_\beta}\beta \tag{7}$$

The roll orientation of the Y and Z axes can be selected[10] so that for zero spin

$$C_{m_\beta} = C_{n_\alpha} \tag{8}$$

$$-C_{m_\alpha} \geq C_{n_\beta} \tag{9}$$

The complex moment then has the form

$$C_m + iC_n = C_{m_0} + iC_{n_0} + [C_{MS_\alpha} - iC_{M_\alpha}]\xi + i\hat{C}_{M_\alpha}\bar{\xi} \tag{10}$$

where

$$\hat{C}_{M_\alpha} = (C_{m_\alpha} + C_{n_\beta})/2 \le 0$$

For nonzero spin, we will assume that Eq. (10) still applies with the possibility that C_{MS_α} can be a function of spin. For the statically stable missiles considered in this paper, C_{M_α} is negative.

For a symmetric missile, $\hat{C}_{M_\alpha}$ is zero and C_{MS_α} is an odd function of spin. (The side moment for this case is usually called a Magnus moment.) The addition of a complex constant $(C_{m_0} + iC_{n_0})$ to the moment of a symmetric missile introduces a trim angle and the possibility of resonance.[2,11,12]

The primary effect of the C_{MS_α} side-moment coefficient on the angular motion of a symmetric missile is on the damping rates and not on the frequencies. The damping rates are assumed to be small compared with the frequencies and will be calculated later as a perturbation of the zero-damped motion. Thus, the side moment will be grouped with the aerodynamic forces and the damping moments and neglected until these perturbations are introduced.

III. Frequencies and Modal Amplitudes

The static moment of Eq. (10) can now be inserted in Eq. (5), the aerodynamic force neglected ($C_D = C_Y = C_Z = 0$) until Sec. VI, and Eq. (4) used to eliminate μ. We obtain

$$\xi'' + if\phi'\xi' - M_r\xi + \hat{M}\bar{\xi} = iA \tag{11}$$

Since Eq. (11) is a fourth-order differential system in the real variables α and β, periodic solutions correspond to two pairs of conjugate roots of a fourth-degree polynomial.

For a symmetric missile with trim, $\hat{M}$ is zero and the solution to Eq. (11) can be written in terms of two rotating complex vectors plus a constant trim vector:

$$\xi = k_1 e^{i\omega_1 s} + k_2 e^{i\omega_2 s} + k_3 \tag{12}$$

where

$$\omega_j = \tfrac{1}{2}\{-f\phi' \pm [(f\phi')^2 - 4M_r]^{1/2}\}$$

$$= -\phi' + \tfrac{1}{2}[P \pm (P^2 - 4M)^{1/2}]$$

$$k_3 = -iA/M_r$$

In the nonrolling aeroballistic coordinates, this becomes

$$\tilde{\xi} = \xi e^{i\phi' s} = k_1 e^{i\phi_1' s} + k_2 e^{i\phi_2' s} + k_3 e^{i\phi' s} \tag{13}$$

where

$$\phi_j' = \omega_j + \phi' = [P \pm (P^2 - 4M)^{1/2}]/2$$

Note that the trim vector k_3 becomes infinite for $M_r = 0$. This is the usual spin-yaw resonance condition and the spin that makes M_r vanish is resonance spin.

Usually ϕ_1' is defined to be the larger frequency and, if we limit the spin to positive values, ω_1 would then correspond to the positive square root and ω_2 to the negative square root. This means that for negative M, ω_2 is always negative and ω_1 is positive for M_r negative and negative for M_r positive. Thus, ω_1 changes sign at resonant spin and is always less in magnitude than ω_2.

Since Eq. (11) contains $\bar{\xi}$, the solution should consist of complex vectors and their conjugates:

$$\xi = k_1 e^{i\omega_1 s} + k_2 e^{i\omega_2 s} + k_3 + k_4 e^{-i\omega_1 s} + k_5 e^{-i\omega_2 s} \tag{14}$$

By direct substitution into Eq. (11), we see that the trim vector takes the very simple form

$$k_3 = -i(M_r A - \hat{M}\bar{A})/(M_r^2 - \hat{M}^2) \tag{15}$$

Resonance now occurs for the two values of spin that make the denominator in Eq. (15) vanish. In terms of pitch and yaw zero-spin-rate frequencies ω_α and ω_β, these spin rates are

$$\phi_l' = \omega_l [f - 1]^{-1/2} \qquad (l = \alpha, \beta) \tag{16}$$

where

$$\omega_\alpha = (-k_t^{-2} C^*_{m_\alpha})^{1/2}$$

$$\omega_\beta = (k_t^{-2} C^*_{n_\beta})^{1/2}$$

Note that the pitch and yaw axes were selected so that

$$\phi_\alpha' \ge \phi_\beta' \tag{17}$$

Direct substitution of Eq. (14) into Eq. (11) yields two pairs of equations, for k_1 and k_4 and for k_2 and k_5. The first pair can be written in the form

$$k_1(C_1 + 2f\phi'\omega_1) - \bar{k}_4\hat{M} = 0 \tag{18}$$

$$-\bar{k}_1\hat{M} + k_4 C_1 = 0 \tag{19}$$

where

$$C_1 = \omega_1^2 - f\phi'\omega_1 + M_r \tag{20}$$

The solution to Eqs. (18) and (19) is

$$k_4 = \hat{M}\bar{k}_1 C_1^{-1} \tag{21}$$

$$C_1^2 + 2f\phi'\omega_1 C_1 - \hat{M}^2 = 0 \tag{22}$$

Similar equations follow for k_2, k_5, C_2, and ω_2.

Since k_1 and k_2 are the only modes present for a symmetric missile, k_4 and k_5 should vanish for $\hat{M} = 0$ and therefore C_j should not be zero for this value of $\hat{M}$. This characteristic allows the selection of the proper roots of the quadratic equation for C_j:

$$C_1 = -f\phi'\omega_1 + \delta[(f\phi'\omega_1)^2 + \hat{M}^2]^{1/2} \tag{23}$$

$$C_2 = -f\phi'\omega_2 + [(f\phi'\omega_2)^2 + \hat{M}^2]^{1/2} \tag{24}$$

where

$$\delta = \begin{cases} 1 \text{ if } \phi' > \phi_\alpha' \\ -1 \text{ if } 0 \le \phi' < \phi_\beta' \end{cases}$$

The frequency equations follow from Eqs. (20), (22), and the corresponding equations for the second frequency. The solutions to the resulting quadratic equations in ω_j^2 are

$$\omega_1 = \delta\left[-\left(\frac{a+\sqrt{b}}{2}\right)^{1/2} + \left(\frac{a-\sqrt{b}}{2}\right)^{1/2}\right] \tag{25}$$

$$\omega_2 = -\left(\frac{a+\sqrt{b}}{2}\right)^{1/2} - \left(\frac{a-\sqrt{b}}{2}\right)^{1/2} \tag{26}$$

where

$$a = \tfrac{1}{2}(f\phi')^2 - M_r$$

$$b = M_r^2 - \hat{M}^2$$

$$= (f-1)^2[(\phi')^2 - (\phi_\alpha')^2][(\phi')^2 - (\phi_\beta')^2]$$

If Eq. (21) and its k_5 counterpart are substituted in Eq. (14) and if the k_j's are expressed in the polar form $K_j \exp(i\phi_{j0})$, the solution motion takes a very simple form in aeroballistic coordinates:

$$\xi = K_1 [e^{i\phi_1} + \hat{M}C_1^{-1} e^{i(2\phi - \phi_1)}] + K_2 [e^{i\phi_2} + \hat{M}C_2^{-1} e^{i(2\phi - \phi_2)}] + k_3 e^{i\phi} \tag{27}$$

for $0 \le \phi' < \phi_\beta'$ or $\phi' > \phi_\alpha'$, where

$$\phi_j = \phi_{j0} + \phi_j' s \qquad (j=1,2)$$

$$\phi_j' = \omega_j + \phi'$$

$$\phi = \phi' s$$

IV. Resonance Region

When the spin rate ϕ' is between ϕ_β' and ϕ_α', b is negative and Eq. (25) is invalid. (It should be noted that Eq. (26) does give a real number for ω_2 although the calculation involves complex numbers.) For this resonance region, the terms with ω_1 in Eq. (14) must be modified:

$$\xi = k_{1R} e^{\lambda s} + k_{4R} e^{-\lambda s} + k_2 e^{i\omega_2 s} + k_5 e^{-i\omega_2 s} + k_3 \tag{28}$$

where

$$\lambda > 0$$

$$k_{jR} = K_{jR} \exp(i\phi_{jR}) \qquad (j=1,4)$$

Substitution of Eq. (28) in Eq. (11) yields two pairs of real equations for the coefficients of $\exp(\lambda s)$ and $\exp(-\lambda s)$.

According to these equations,

$$\tan\phi_{1R} = -\tan\phi_{4R} = -f\phi'\lambda / (\lambda^2 - M_r - \hat{M}) \tag{29}$$

$$\lambda^4 + 2a\lambda^2 + b = 0 \tag{30}$$

For an almost-symmetric missile ($\hat{M} \ll M$), $b \ll a^2$, and

$$\lambda \doteq (\hat{M}^2 - M_r^2)^{1/2} / (f\phi') = \sqrt{-b} / (f\phi') \tag{31}$$

$$\tan\phi_{1R} \doteq \{[(\phi')^2 - (\phi_\beta')^2] / [(\phi_\alpha')^2 - (\phi')^2]\}^{1/2} \tag{32}$$

According to these approximations, λ vanishes for spin on the boundaries of the resonance region and is a maximum near the center ($M_r = 0$). The planes of the exponentially growing and decaying angles are near the pitch plane for spin near ϕ_α' and are near the yaw plane for spin near ϕ_β'. For maximum growth/decay rates, the planes of the growing angle and the decaying angle are perpendicular to each other and both planes bisect the angle between the pitch and yaw planes.

V. C_j and ω_j for Almost-Symmetric Missiles

We will now obtain special relations for an almost-symmetric missile for which $\hat{M}$ is much smaller than M. For this case, the radical in Eqs. (23) and (24) for C_j can be expanded for three regions of spin—near zero, near resonance, and otherwise. These expansions yield the results in Table 1.

Far from resonance, $M_r^2 \gg \hat{M}^2$ and a good estimate of the frequencies can be obtained from the approximation

$$\sqrt{b} \doteq \delta (M_r - \hat{M}^2 / 2M_r) \tag{33}$$

Equations (25) and (26) can now be expanded in terms of $\hat{M}$ to yield the frequencies given in Table 2. According to these equations, very good approximations to the frequencies of an almost-symmetric missile are the symmetric frequencies of a missile with the average moment coefficient C_{M_α} of the almost-symmetric missile. This is correct to within a term in $\hat{M}^2$. Note that the coefficients g_j are unbounded for zero spin but only g_1 has difficulty near resonance. (g_2 has the limit $-[(f\phi')^{-3}]/2$ as $M_r \to 0$.)

Table 1 Approximations for C_j

ϕ'	C_1	C_2
$0 \le \phi' < \epsilon$	$\hat{M} - f\phi'\omega_1$	$-M - f\phi'\omega_2$
$\phi_\beta' - \epsilon < \phi' \le \phi_\beta'$	$\hat{M} - f\phi'\omega_1$	$-2f\phi'\omega_2$
$\phi_\alpha' \le \phi' < \phi_\alpha' + \epsilon$	$-M - f\phi'\omega_1$	$-2f\phi'\omega_2$
otherwise	$-2f\phi'\omega_1$	$-2f\phi'\omega_2$

where

$$0 < \epsilon \ll \phi_\alpha' - \phi_\beta'$$

Table 2 Frequencies for an almost-symmetric missile

ϕ'	ϕ_1'	ϕ_2'
$0 \le \phi' < \epsilon$	$\omega_\beta + \phi' + h(\phi')^2$	$-\omega_\alpha + \phi' + h(\phi')^2$
$0 \ll \phi' \ll \phi_\beta'$	$\phi_{1S}' + g_1 \hat{M}^2$	$\phi_{2S}' + g_2 \hat{M}^2$
$\phi_\beta' - \epsilon < \phi' \le \phi_\beta'$	$\phi' + [M_r^2 - \hat{M}^2]^{1/2} (f\phi')^{-1}$	$\phi_{2S}' + g_2 \hat{M}^2$
$\phi_\beta' \le \phi' \le \phi_\alpha'$	ϕ'	$\phi_{2S}' + g_2 \hat{M}^2$
$\phi_\alpha' \le \phi' < \phi_\alpha' + \epsilon$	$\phi' - [M_r^2 - \hat{M}^2]^{1/2} (f\phi')^{-1}$	$\phi_{2S}' + g_2 \hat{M}^2$
$\phi_\alpha' \ll \phi'$	$\phi_{1S}' + g_1 \hat{M}^2$	$\phi_{2S}' + g_2 \hat{M}^2$

where

$0 < \epsilon \ll \phi_\alpha' - \phi_\beta'$ $\qquad g_1 = -\omega_{2S} [2f\phi' M_r (\omega_{1S} - \omega_{2S})]^{-1}$

$h = (-M)^{1/2} f / 4\hat{M}$ $\qquad g_2 = \omega_{1S} [2f\phi' M_r (\omega_{1S} - \omega_{2S})]^{-1}$

Table 2 also gives the frequencies near resonance and near zero spin, as derived from Eqs. (25) and (26). According to this table, ϕ_1' at zero spin is the zero-spin yaw frequency and ϕ_1' increases as the spin increases. At frequencies well away from zero or ϕ_β', ϕ_1' is well approximated by ϕ_{1S}'. In the resonance region, it is the spin frequency, and at spins well above the resonance region, it is once again well approximated by the symmetric-missile fast frequency.

ϕ_2' has a much simpler behavior. At zero spin, it is the negative of the zero-spin pitch frequency and increases as spin increases. Well away from zero spin, it is well approximated by ϕ_{2S}'.

In Table 3, the various approximations for C_j are used to construct expressions for the various motions for an almost symmetric missile with no trim moment ($k_3 = 0$). This catalog of motion starts at zero spin with the expected Lissajous figure formed by combining the pitching and yawing motion with different frequencies (case A). For slow spin, the case A motion becomes more complicated.

Case B is the general case for spin well away from zero or resonance. It is a four-mode motion with two modes quite similar to those associated with the symmetric missile epicyclic motion with frequencies ϕ_1' and ϕ_2'. The two additional modes have much smaller amplitudes that go to zero as the missile becomes symmetric ($\hat{M} \to 0$). Their frequencies have the rather odd form $2\phi' - \phi_j'$.

As the spin approaches the resonance region, ϕ_1' and $2\phi' - \phi_1'$ become equal and the motion associated with K_1 changes character. The amplitude of the $2\phi' - \phi_1'$ mode approaches the amplitude of the ϕ_1' mode (case C). In the resonance region (case D), the two modes associated with ϕ_1' both rotate at the spin rate, but one mode grows exponentially and the other decays exponentially. The exponent λ depends on the

Table 3 Aeroballistic motion

A. $0 \le \phi' < \epsilon$

$$\xi = K_1 \left[e^{i\phi_1} + [1 - f\phi'\omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - [1 + f\phi'\omega_2 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_2)} \right]$$

B. $0 \ll \phi' \ll \phi'_\beta, \phi' \gg \phi'_\alpha$

$$\xi = K_1 \left[e^{i\phi_1} - \hat{M}(2f\phi'\omega_1)^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M}(2f\phi'\omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

C. $\phi'_\beta - \epsilon < \phi' < \phi'_\beta$

$$\xi = K_1 \left[e^{i\phi_1} + [1 - f\phi'\omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M}(2f\phi'\omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

D. $\phi'_\beta < \phi' < \phi'_\alpha$ **(Resonance Region)**

$$\xi = e^{i\phi} \left[K_{1R} e^{\lambda s + i\phi_{1R}} + K_{4R} e^{-\lambda s - i\phi_{1R}} \right] + K_2 \left[e^{i\phi_2} - \hat{M}(2f\phi'\omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

E. $\phi'_\alpha < \phi' < \phi'_\alpha + \epsilon$

$$\xi \neq K_1 \left[e^{i\phi_1} - [1 + f\phi'\omega_1 \hat{M}^{-1}]^{-1} e^{i(2\phi - \phi_1)} \right] + K_2 \left[e^{i\phi_2} - \hat{M}(2f\phi'\omega_2)^{-1} e^{i(2\phi - \phi_2)} \right]$$

F. $\phi' \gg \phi'_\alpha$

Same as case B, where

$$\omega_j = \phi'_j - \phi'$$

$$\lambda = (f-1)(f\phi')^{-1} \left[[(\phi')^2 - (\phi'_\beta)^2][(\phi'_\alpha)^2 - (\phi')^2] \right]^{1/2}$$

$$\tan\phi_{1R} = \left[[(\phi')^2 - (\phi'_\beta)^2][(\phi'_\alpha)^2 - (\phi')^2] \right]^{1/2}$$

and where K_{1R} and K_{4R} are parameters determined by initial conditions.

size of the resonance region and how far the spin rate is from the boundaries. Case E is the motion for spins slightly above the resonance spin region.

In order to show this spectrum of motions in a more specific way, the frequencies, modal amplitudes, and resonance damping rate have been computed for an almost-symmetric missile with $\hat{C}_{M_\alpha} = 0.095 C_{M_\alpha}$ and $f = 1.9$. For these values,

$$\phi'_\alpha / \phi'_\beta = 1.10 \qquad \omega_\alpha / \phi'_\beta = 1.04 \qquad \omega_\beta / \phi'_\beta = 0.95$$

Figures 1 and 2 give the actual frequencies as functions of ϕ'/ϕ'_β and compare them with the corresponding frequencies for a symmetric missile with moment coefficient C_{M_α}. ϕ'_1 is essentially ϕ'_{1S} except near zero spin and resonance and ϕ'_2 is essentially ϕ'_{2s} except near zero spin. At zero spin, ϕ'_1 and ϕ'_2 reduce to ω_β and $-\omega_\alpha$, respectively.

Figure 3 shows the variation of the "asymmetric" modal amplitudes K_4 and K_5 in comparison with the corresponding "symmetric" modal amplitudes K_1 and K_2. K_4 is quite small except near zero spin and near resonance and K_5 is quite small except near zero spin. It is important to note that K_4 is always a larger fraction of K_1 than K_5 is of K_2. This is due to the presence of ω_j in C_j and the fact that ω_1 is always smaller in amplitude than ω_2. Thus we would expect to observe the presence of the $2\phi' - \phi'_1$ mode before the $2\phi' - \phi'_2$ mode has a noticeable effect.

Finally, the resonance damping rate is plotted in Figure 4. The peak value of 0.047 corresponds to about a 30% increase in amplitude during one revolution.

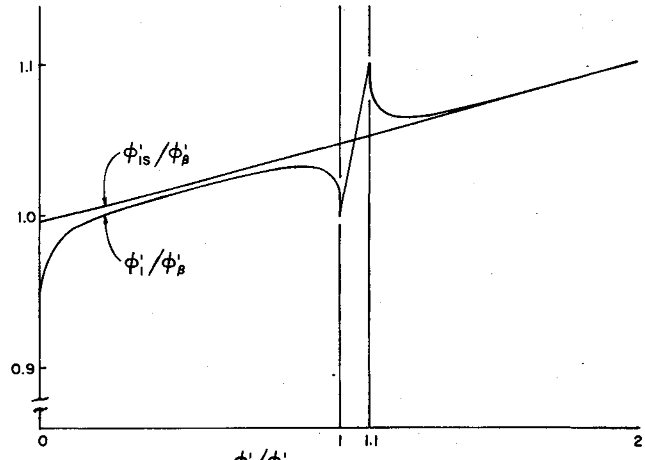

Fig. 1 Frequencies ϕ'_1/ϕ'_β and ϕ'_{1S}/ϕ'_β vs spin rate ϕ'/ϕ'_β.

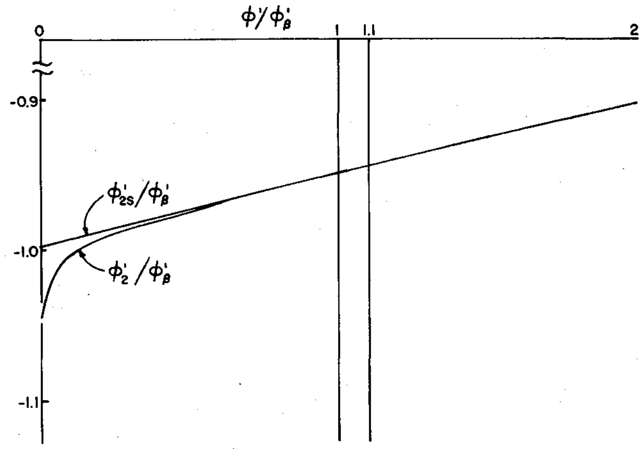

Fig. 2 Frequencies ϕ'_2/ϕ'_β and ϕ'_{2S}/ϕ'_β vs spin rate ϕ'/ϕ'_β.

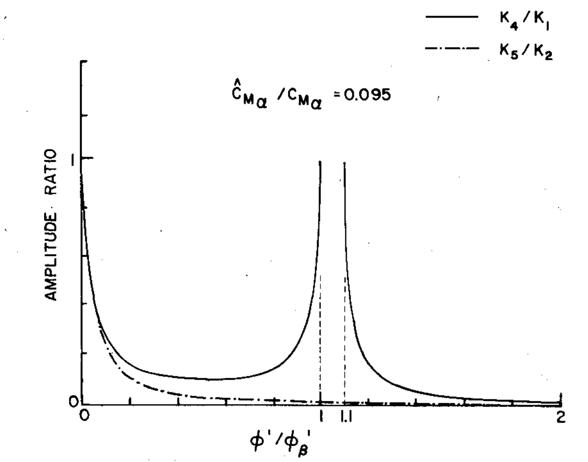

Fig. 3 Amplitude ratios K_4/K_1 and K_5/K_2 vs spin rate ϕ'/ϕ'_β.

VI. Effect of Linear Forces and Damping Moments

In order to complete our study of an almost-symmetric missile, we will now assume asymmetric normal force and damping moments in addition to the side moment we omitted in Sec. II. These terms are generalizations of those of Ref. 9. The new coefficients are defined in Table 4. The assumed force and moment can be inserted in Eqs. (4) and (5) and μ eliminated between the resulting equations. (Due to size

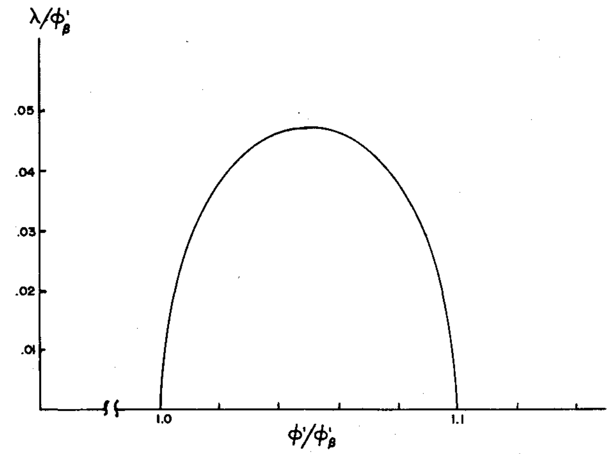

Fig. 4 Resonance damping rate λ/ϕ_β' vs spin rate ϕ'/ϕ_β'.

considerations, products of C^* terms are ignored.) We obtain

$$\xi'' + (H+if\phi')\xi' - [M_r + i(M_S - \phi' H)]\xi = iA - \hat{H}\bar{\xi}' - [\hat{M} + i(\hat{M}_S - \phi'\hat{H})]\bar{\xi} \tag{34}$$

The solution is assumed to have the form

$$\xi = k_1 e^{(\lambda_1 + i\omega_1)s} + k_2 e^{(\lambda_2 + i\omega_2)s} + k_3 + k_4 e^{(\lambda_1 - i\omega_1)s} + k_5 e^{(\lambda_2 - i\omega_2)s} \tag{35}$$

The trim term, k_3, can be obtained by direct substitution in Eq. (34):

$$k_3 = \frac{-i\{[M_r - i(M_S - \phi' H)]A - [\hat{M} + i(\hat{M}_S - \phi'\hat{H})]\bar{A}\}}{M_r^2 - \hat{M}^2 + (M_S - \phi' H)^2 - (\hat{M}_S - \phi'\hat{H})^2} \tag{36}$$

The damping exponent has the following simple approximation for spins far from zero and the resonance region[10]:

$$\lambda_1 = \lambda_{1S} + \hat{M}\hat{H}/f\phi'(2\phi_1' - P) \tag{37}$$

Similarly,

$$\lambda_2 = \lambda_{2S} + \hat{M}\hat{H}/f\phi'(2\phi_2' - P) \tag{38}$$

Table 4 Aerodynamic coefficients

$C_{N_\alpha} = -(C_{Y_\beta} + C_{Z_\alpha})/2$	$\hat{C}_{N_\alpha} = -(C_{Y_\beta} - C_{Z_\alpha})/2$
$C_{M_\alpha} = (C_{m_\alpha} - C_{n_\beta})/2$	$\hat{C}_{M_\alpha} = (C_{m_\alpha} + C_{n_\beta})/2$
$C_{M_{\dot\alpha}} = (C_{m_{\dot\alpha}} - C_{n_{\dot\beta}})/2$	$\hat{C}_{M_{\dot\alpha}} = (C_{m_{\dot\alpha}} + C_{n_{\dot\beta}})/2$
$C_{M_q} = (C_{m_q} + C_{n_r})/2$	$\hat{C}_{M_q} = (C_{m_q} - C_{n_r})/2$
$C_{MS_\alpha} = (C_{m_\beta} + C_{n_\alpha})/2$	

where

$C_Y = C_{Y_\beta}\beta$

$C_Z = C_{Z_\alpha}\alpha$

$C_m = C_{m_0} + C_{m_\beta}\beta + C_{m_\alpha}\alpha + C_{m_{\dot\alpha}}(\alpha' + \phi'\beta) + C_{m_q}(ql/V)$

$C_n = C_{n_0} + C_{n_\beta}\beta + C_{n_\alpha}\alpha + C_{n_{\dot\beta}}(\beta' - \phi'\alpha) + C_{n_r}(rl/V)$

Thus the damping exponent differs only a little from the damping for a symmetric missile with the average moment coefficients.

The bounds of the resonance region are set by the zeros of the denominator of Eq. (36). If we neglect the side-moment coefficient, this condition has the simple form:

$$M_r^2 - \hat{M}^2 + (\phi')^2 F = 0 \tag{39}$$

where

$$F = [H - (2-f)(C_{N_\alpha}^* - C_D^*)]^2 - (\hat{H} - f\hat{C}_{N_\alpha}^*)^2$$

If we denote the damping-modified resonance spins as $\phi_{\beta R}'$ and $\phi_{\alpha R}'$ and assume F to be small, the modified resonance spins can be computed as perturbations of the zero-damping resonance spins ϕ_β' and ϕ_α':

$$\phi_{\beta R}' = [1 - F/4(f-1)\hat{M}]\phi_\beta' \tag{40}$$

$$\phi_{\alpha R}' = [1 + F/4(f-1)\hat{M}]\phi_\alpha' \tag{41}$$

Since $\hat{M}$ was selected to be negative, a positive F reduces the size of the resonance region and negative F increases it.

VII. Conclusion

The primary value of this work is the insight given an engineer in interpreting the numerical calculation or the flight measurements of the angular motion of an almost-symmetric missile. Although the relations of Secs. II-IV are valid for any degree of asymmetry, the simplicity of the results for an almost-symmetric missile should prove a great help to an engineer familiar with the classical tricyclic theory of a slightly asymmetric missile.

The four specific characteristics of the motion of an almost-symmetric missile are:

1) The general motion is well approximated by a symmetric missile with average coefficients.

2) Far from zero spin or resonance spin rates, the first observable modification of the usual tricyclic motion for an almost-symmetric missile is the appearance of a $2\phi' - \phi_1'$ frequency, followed by the appearance of a $2\phi' - \phi_2'$ frequency as the asymmetry becomes greater.

3) Near zero spin, both of these additional frequencies have substantial amplitudes, and near resonance, the $2\phi' - \phi_1'$ frequency has a substantial amplitude.

4) For spin in the resonance region, large trims and exponential undamping are possible.

References

[1] Ward, L.K. and Mansfield, A.C., "Dynamic Characteristics of a 9-Degree Cone With and Without Asymmetries at Mach Number 10," Arnold Engineering Development Center, TR-70-1, March 1970.

[2] Nicolaides, J.D., "On the Free Flight Motion of Missiles Having Slight Configurational Asymmetries," U.S. Army Ballistic Research Laboratories Report No. 858, AD 26405, June 1953. (See also IAS Preprint 395, 1952.)

[3] Walchner, O., Sawyer, F.M., and Yelmgren, K.E., "Aerodynamics of a Slender Cone with Asymmetric Nose Bluntness at Mach 14," *AIAA Journal*, Vol. 10, Aug. 1972, pp. 1121-1122.

[4] Intrieri, P.F., Kirk, D.B., Chapman, G.T., and Terry, J.E., "Ballistic Range Tests of Ablating and Nonablating Slender Cones," *AIAA Journal*, Vol. 8, March 1970, pp. 558-564.

[5] Jaffe, P., "Nonplanar Tests Using Wind Tunnel Free-Flight Technique," *Journal of Spacecraft and Rockets*, Vol. 10, July 1973, pp. 435-442.

[6]Phillips, W.H., "Effect of Steady Rolling on Longitudinal and Directional Stability," NACA Technical Note No. 1627, June 1948.

[7]Hodapp, A.E., Jr., "Effects of Unsymmetrical Stability Derivative Characteristics on Re-Entry Vehicle Trim Angle Behavior," *Journal of Spacecraft and Rockets,* Vol. 11, May 1974, pp. 300-307.

[8]Hodapp, A.E., Jr., "Effects of Unsymmetrical Stability Derivative Characteristics on Re-Entry Vehicle Transient Angular Motion," *Journal of Spacecraft and Rockets,* Vol. 13, Feb. 1976, pp. 82-90.

[9]Murphy, C.H., "Free-Flight Motion of Symmetric Missiles," U.S. Army Ballistic Research Laboratories Report No. 1216, AD 442757, July 1963.

[10]Murphy, C.H., "Angular Motion of Spinning Almost-Symmetric Missiles," U.S. Army Ballistic Research Laboratory Technical Report ARBRL-TR-02121, AD A063538, Nov. 1978.

[11]Nelson, R.L., "The Motions of Rolling Symmetrical Missiles Referred to Body-Axis System," NACA Technical Note No. 1627, Nov. 1956.

[12]Price, D.A., Jr., and Ericsson, L.E., "A New Treatment of Roll-Pitch Coupling for Ballistic Re-Entry Vehicles," *AIAA Journal,* Vol. 8, Sept. 1970, pp. 1608-1615.